$$\sum_{i=1}^{i=n} (x_i \cos\alpha + y_i \sin\alpha)^2 = \text{Minimum.}$$

Equating to zero the derivative as to α,

$$\sum_{i=1}^{i=n} (y_i^2 - x_i^2) \sin\alpha \cos\alpha = \sum_{i=1}^{i=n} x_i y_i (\sin^2\alpha - \cos^2\alpha).$$

Let

$$\frac{y_i}{x_i} = \theta_i,$$

$$\tan 2\alpha = \frac{\sum_{i=1}^{i=n} \sin 2\theta_i}{\sum_{i=1}^{i=n} \cos 2\theta_i}.$$

The origin shall be O, the point (1, 0) shall be O'. We construct a unit circle about O' as center. Let OP_i meet this circle again in Q_i. The coördinates of Q_i in the system *where O' is origin* will be $(\cos 2\theta_i, \sin 2\theta_i)$. Let $\bar{Q}$ be the center of gravity of the points Q_i. The line $O'\bar{Q}$ shall meet the circle in R, that is to say, R is the intersection which lies on the same side of the diameter OO' as does $\bar{Q}$. Then OR is the required line. See Fig. 2.

COMPUTATION FORMULA FOR THE PROBABILITY OF AN EVENT HAPPENING AT LEAST C TIMES IN N TRIALS.

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Many problems in biometry, radioactivity, etc., require for their quantitative solution a convenient formula for computing the probability of an event happening at least c times in n trials; the probability p of its happening in one trial being known. Several relatively simple formulas are given by Poisson in the third chapter of his *Recherches sur la Probabilité des Jugements*, availing himself of a method developed by Laplace in the *Théorie Analytique des Probabilités*. But these formulas explicitly exclude the case where p is very small and $c = np + r$, r not being small compared with np. For this range of values a problem connected with my engineering work forced me early in 1908 to develop the formulas* (5) and (6) given below.

Let P = the required probability, $\alpha = np$, $s = (n - c)/(c + 1)$,

$$F(x) = \frac{2c(1-x)x^2 + (n-s)sx^4}{[c - (n-s)x]^4}$$

* These formulas have been used for the construction of curves which are in constant use in the Engineering Department of the American Telephone and Telegraph Company. They are published in the hope that others will find them helpful.

and take as a starting point Poisson's exact equation

$$P = \frac{\int_\sigma^\infty Y dy}{\int_0^\infty Y dy}, \qquad \text{where} \qquad Y = \frac{y^{n-c}}{(1+y)^{n+1}} \qquad \text{and} \qquad \sigma = \frac{1-p}{p}.$$

Transforming the numerator by the change of variable $y = (1/x) - 1$ and substituting for the integral in the denominator its well-known value, gives

$$P = \frac{n!}{(c-1)!\,(n-c)!} \int_0^p x^{c-1}(1-x)^{n-c} dx, \tag{1}$$

or, as

$$c = np + r > (n-s)p,$$

we may write

$$P = \frac{n!}{(c-1)!\,(n-c)!} \int_0^p \frac{\partial}{\partial x} \{x^c(1-x)^{n-c-s}\} \frac{(1-x)^{s+1}}{[c-(n-s)x]} dx.$$

By partial integration

$$P = \frac{n!}{(c-1)!\,(n-c)!} \left\{ \frac{p^c(1-p)^{n-c+1}}{[c-(n-s)p]} - (n-s)s \int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2} dx \right\}. \tag{2}$$

Also, by partial integration

$$\begin{aligned} \int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2} dx &= \int_0^p \frac{\partial}{\partial x} \{x^c(1-x)^{n-c-s}\} \frac{x^2(1-x)^{s+1}}{[c-(n-s)x]^3} dx \\ &= \frac{p^{c+2}(1-p)^{n-c+1}}{[c-(n-s)p]^3} - \int_0^p x^{c-1}(1-x)^{n-c} F(x) dx. \end{aligned} \tag{3}$$

Now, since $F(x)$ remains finite, does not change sign and is a maximum for $x = p$ as x varies from 0 to p, provided $p < 2/3$,

$$\int_0^p x^{c-1}(1-x)^{n-c} F(x) dx = \lambda F(p) \int_0^p x^{c-1}(1-x)^{n-c} dx, \qquad 0 < \lambda < 1. \tag{4}$$

By (2), (3), (4) and the expressions for s and $F(p)$ we obtain

$$\begin{aligned} P = \frac{n!}{c!\,(n-c)!} p^c(1-p)^{n-c} &\left\{ \frac{(1-p)(c+1)}{c+1-(n+1)p} \right\} \\ &\times \left\{ 1 - \frac{(\alpha+p)(\alpha-cp)}{c(c+1-\alpha-p)^2} \right\} \left\{ \frac{1}{1-\lambda\varphi} \right\} \end{aligned} \tag{5}$$

where

$$\varphi = \frac{(\alpha+p)(\alpha-cp)\{2(c+1)^2(1-p)+(\alpha+p)(\alpha-cp)\}}{c^2(c+1-\alpha-p)^4} < \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha-p)^4}.$$

For p not greater than 1/30 and P not greater than .01 I found that the values of c and α $(= np)$ are such that φ is negligible. For this range of values, therefore, the factor $1/(1 - \lambda\varphi)$ in the right-hand member of (5) was ignored and a convenient formula for computing P thereby obtained. After plotting several curves it became evident that for a given value of P the corresponding value of c did not change appreciably with changes in the values of p and n, provided the product α $(= np)$ was kept constant. The following formulas were therefore used.

In (5) let n and p approach the limits ∞ and 0 respectively while α remains constant. Then in the limit

$$(6) \qquad \begin{aligned} P &= \left\{\frac{1}{1-\lambda\psi}\right\}\left\{\frac{\alpha^c\epsilon^{-\alpha}}{c!}\right\}\left\{\frac{c+1}{c+1-\alpha}\right\}\left\{1-\frac{\alpha^2}{c(c+1-\alpha)^2}\right\}, \\ \psi &= \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha)^4}. \end{aligned}$$

Let P_0, P_1 be the values of P obtained from (6) by taking λ equal to 0 and 1 respectively. Then

$$(7) \qquad P_0 < P < P_1.$$

Therefore $(P_1 + P_0)/2$ may be taken as an approximate value of P provided $(P_1 - P_0)/2$ is small compared with P_0. This will be the case if P is not greater than .01 and α/c small. As α/c increases the discrepancy between P and $(P_1 + P_0)/2$ increases, but not indefinitely. This may be proved as follows.

For a given value of P we know (Bernoulli-Laplace Theorem) that α/c approaches the limit 1 as α and c approach ∞. Also, for α and c very large, one of Poisson's formulas is a very close approximation and reduces to

$$(8) \qquad P = \frac{2}{\sqrt{\pi}}\int_\kappa^\infty \epsilon^{-t^2}dt$$

where κ, the lower limit of the integral, is given by the equation

$$\begin{aligned} \frac{\kappa^2 + (c-1-\alpha)}{(c-1)} &= -\log\left\{\frac{\alpha}{c-1}\right\} = -\log\left\{1-\frac{c-1-\alpha}{c-1}\right\} \\ &= \frac{(c-1-\alpha)}{(c-1)} + \frac{(c-1-\alpha)^2}{2(c-1)^2} + \frac{(c-1-\alpha)^3}{3(c-1)^3} + \cdots, \end{aligned}$$

or, for c very large and $\alpha/c \doteq 1$

$$(9) \qquad \kappa^2 = \frac{(c-1-\alpha)^2}{2(c-1)} = \frac{(c+1-\alpha)^2}{2\alpha},$$

but for $\alpha/c \doteq 1$

$$(10) \qquad \psi = \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha)^4} = \frac{3\alpha^2}{(c+1-\alpha)^4} = \frac{3}{(2\kappa^2)^2}.$$

Therefore, by (10) and any of the well-known tables giving the values of P and κ which satisfy equation (8), we obtain the following figures showing the limits (as $c \doteq \infty$ and $\alpha/c \doteq 1$) of the possible error incurred by taking $P = (P_1 + P_0)/2$.

P	κ^2	ψ	$(P_1 - P_0)/2P_0$
.01	2.72	.101	.056
.001	4.80	.032	.017
.0001	6.92	.016	.008

The short table which follows gives some of the pairs of values of c and α satisfying the three equations $P = .01$, $P = .001$ and $P = .0001$ for the limiting case of $p \doteq 0$, $n \doteq \infty$.

	$P = .0001$		$P = .001$		$P = .01$	
c	α	α/c	α	α/c	α	α/c
1	.0001	.0001	.0010	.001	.010	.010
2	.0142	.0071	.0454	.023	.149	.075
3	.0862	.0289	.191	.064	.436	.145
4	.232	.0580	.429	.107	.823	.206
5	.444	.0888	.739	.148	1.28	.256
6	.714	.119	1.11	.185	1.79	.298
7	1.03	.147	1.52	.217	2.33	.333
8	1.39	.174	1.97	.246	2.91	.364
9	1.78	.198	2.45	.272	3.51	.390
10	2.20	.220	2.96	.296	4.13	.413
12	3.11	.259	4.04	.337	5.43	.452
14	4.11	.293	5.20	.371	6.78	.484
16	5.17	.323	6.41	.401	8.18	.512
18	6.28	.349	7.66	.426	9.62	.534
20	7.44	.372	8.96	.448	11.1	.555
30	13.7	.457	15.9	.530	18.7	.623
40	20.6	.515	23.3	.582	26.8	.670
60	35.4	.590	38.0	.648	43.5	.725
80	50.9	.636	55.2	.690	60.7	.759
100	67.0	.670	71.9	.719	78.2	.782
140	100.2	.716	106.3	.760	114.0	.814
180	134.3	.746	141.4	.786	150.3	.835
200	151.6	.758	159	.795	169	.845
300	240	.800	249	.830	261	.870
400	330	.825	341	.852	355	.887
500	421	.842	434	.868	450	.900
600	513	.855	527	.878	545	.908
700	606	.866	621	.887	640	.914
800	699	.874	715	.894	736	.920
900	793	.881	810	.900	832	.924
1000	887	.887	905	.905	928	.928

BOOK REVIEWS.

W. H. BUSSEY, Chairman of the Committee.

Academic Algebra. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn & Co., Boston, 1913. iv + 442 pages. $1.20.

This book is the second of the Wentworth-Smith series. In it the authors have attempted to provide a high school course which shall cover the topics